Boundary-layer drag in three-dimensional supersonic flow

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A general theorem for drag is given according to which the boundary-layer drag of a body is equal to the inviscid drag of the displacement surface together with a term which is given as an integral involving the 'streamwise' momentum and displacement thicknesses taken round the trailing edge. A less accurate result for thin slender wings is that the boundary-layer drag is equal to the line integral

$$\rho_\infty U_\infty^2 \int \Theta \, d\sigma$$

taken round the trailing edge, where Θ is the streamwise momentum thickness. This result leads to the possibility of finding boundary-layer drag by means of a traverse round the trailing edge. The extension of the results to wings with swept trailing edges is also given.

1. Introduction

The effect of the boundary layer on the drag of wings and bodies is usually divided into two parts, which have been given the names 'boundary-layer pressure drag' and 'skin-friction drag'. The first is due to the effect of the boundary layer on the pressure distribution over the surface of the body. It is possible for this to be negative; for instance if there is considerable thickening of the boundary layer at the rear, these might be sufficient increase of pressure over backwards facing surfaces to give a resultant forward thrust. The remainder of the drag is then skin-friction drag. We shall not here divide the drag in this way but have preferred instead to take one part of the total drag to be the drag on a fictitious body known as the displacement surface; the other part will be what is left, which will be mainly, but not entirely, skin-friction drag. Such a division is in any case artificial; what we require to know is the total drag due to boundary-layer effects.

In boundary-layer theory it is usually accepted that the external flow behaves as though it were over a distorted body thickened by an amount δ^*, the displacement thickness and it is this body which we shall call the displacement surface. Lighthill (1958) has shown how δ^* can be calculated in three-dimensional flow.

We take Cartesian co-ordinates Ox, Oy and Oz with O at the pointed apex of the body and Ox parallel to the free-stream direction. We take Δ^* and Θ as displacement and momentum thicknesses based on x-components of velocity at the trailing edge and we write $H = \Delta^*/\Theta$. We first prove a general theorem for pointed bodies in supersonic flow. According to this theorem the drag of a body is equal to the drag of a body coinciding with the displacement surface together with a term qN, where N is given by

$$N = 2\int \frac{\rho_D u_D^2}{\rho_\infty U_\infty^2}\Theta\, d\sigma + 2\int \frac{\rho_D u_D u_D'}{\rho_\infty U_\infty^2}(\Delta^* - \delta^*)\, d\sigma - \int c_p \delta^*\, d\sigma + O(\delta^2), \tag{1}$$

in which the integrals are taken round the trailing edge, and

$$q = \tfrac{1}{2}\rho_\infty U_\infty^2, \quad u_D' = u_D - U_\infty. \tag{2}$$

The subscript D refers to values at the trailing edge due to inviscid flow over the undistorted body, and δ is the boundary-layer thickness and c_p is the pressure coefficient at the trailing edge for the undistorted body.

If the body is slender these integrals may be simplified. Taking the maximum semi-span to be s we find that

$$N = N' + E', \tag{3}$$

where

$$N' = 2\int \Theta\left(\frac{u_D}{U_\infty}\right)^{2+H-M^2} d\sigma + O(\delta s^5 \log^2 s), \tag{4}$$

$$E' = \int \frac{v_D^2 + w_D^2}{U_\infty^2}\Delta^*\, d\sigma + O(\delta s^5 \log^2 s). \tag{5}$$

Here M is the Mach number at infinity. If the body is thin and of maximum thickness t the order-of-magnitude terms are replaced by $O(\delta s^3 t^2 \log^2 s)$ and E' itself is of order $\delta s^3 t^2$ and may be omitted. Terms of order δ^2 are neglected in all cases.

We also give a less precise but greatly simplified version of the general theorem, obtained by neglecting terms of order $\delta s^3 \log s$ or $\delta s^2 t \log s$ as the case may be. The result is: The drag of a body is equal to the inviscid drag of the body together with an amount qN_1, where

$$N_1 = 2\int \Theta\, d\sigma \tag{6}$$

taken round the trailing edge.

Of course the success of this method of determining drag depends on being able to calculate the boundary layer, the shape of the corresponding displacement surface and the inviscid flow over this surface. Fortunately it appears that, under certain conditions at least, the simplified form of the general theorem is sufficiently accurate. When this is so that contribution to drag is simply qN_1. This involves only the calculation of Θ at the trailing edge.

The simplified form suggests the possibility of determining the boundary-layer drag by means of a traverse at the trailing edge. There is of course in general a shock at the trailing edge, but if the boundary layer is turbulent the upstream influence of this shock is usually considered to be limited to one or two boundary-layer thicknesses and it should not be too difficult to make the traverse close to the trailing edge yet sufficiently far upstream of the shock for it to have no influence. In some pictures the shock seems to start slightly downstream of the trailing edge, so that it should be possible to make the traverse quite near to the trailing edge.

For simplicity we have chosen a delta wing with rhombic cross-sections and parabolic biconvex centre-section at zero lift for our example, and the boundary layer has been assumed to grow over this wing in the same way as that over a flat plate. It is probable that this assumption is not too inaccurate for thin wings at zero lift, except perhaps near to the centre-line of the wing. On this assumption the expressions (4) and (5) can be worked out in full, and it is indeed found that only N_1 gives a significant contribution to the total boundary-layer drag. Thus the simplified form of the theorem is shown to be adequate in this case.

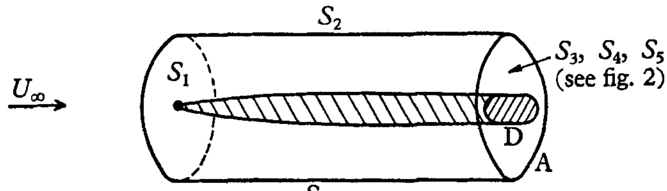

FIGURE 1. The control surface.

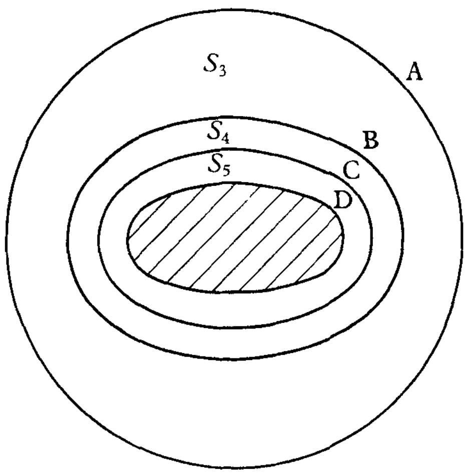

FIGURE 2. Section by the plane $x = 1$. A, Edge of control cylinder; B, edge of boundary layer; C, edge of displacement surface; D, edge of body.

2. Momentum balance

Following Ward (1949) we surround the body by a cylinder of radius r, with plane ends normal to the stream at the leading and trailing edges. For the moment we suppose that the back of the body lines in the latter plane. Taking U_∞ to be the velocity at infinity, $\bar{\phi}$ the disturbance potential due to a distorted body, namely one thickened by an amount δ^*, where δ^* is the extra thickness of the displacement surface, we may write the velocity components $\bar{u}$, $\bar{v}$ and $\bar{w}$

$$\bar{u} = U_\infty\left(1+\frac{\partial\bar{\phi}}{\partial x}\right), \quad \bar{v} = U_\infty\frac{\partial\bar{\phi}}{\partial y}, \quad \bar{w} = U_\infty\frac{\partial\bar{\phi}}{\partial z}, \tag{7}$$

outside the boundary layer. A bar over any quantity signifies that its value for the distorted body is to be taken.

We shall write $$\bar{u}' = \bar{u} - U.$$

We denote by $S_1, S_2, \ldots,$ certain surfaces as shown in figures 1 and 2. Figure 2 is an end-on view of the section by the plane at the trailing edge. In this figure the curve B is the section of the edge of the boundary layer, C is the section of the displacement surface and D is the bounding curve of the blunt base of the body. We denote the value of the x-component of the velocity inside the boundary layer by u_b and the corresponding density by ρ_b.

Conservation of mass through the cylinder gives

$$\int_{S_1} \rho_\infty U_\infty dS - \int_{S_2} \bar{\rho} U_\infty \frac{\partial \bar{\phi}}{\partial r} dS - \int_{S_3} \bar{\rho} U_\infty \left(1 + \frac{\partial \bar{\phi}}{\partial x}\right) dS - \int_{S_4+S_5} \rho_b u_b dS = 0. \quad (8)$$

The drag force on the body and the component in the x-direction of the pressure forces on the cylinder will together balance the flux of x momentum of the fluid leading the cylinder. Hence the drag $\bar{D}$ is given by

$$\bar{D} = \int_{S_1} (p_\infty + \rho_\infty U_\infty^2) dS - \int_{S_2} \bar{\rho} U_\infty^2 \left(1 + \frac{\partial \bar{\phi}}{\partial x}\right) \frac{\partial \bar{\phi}}{\partial r} dS$$
$$- \int_{S_3} \left\{\bar{p} + \bar{\rho} U_\infty^2 \left(1 + \frac{\partial \bar{\phi}}{\partial x}\right)^2\right\} dS - \int_{S_4+S_5} (p_b + \rho_b u_b^2) dS - p_B S(D). \quad (9)$$

In this equation p_b is the pressure inside the boundary layer, p_B is the mean base pressure and $S(D)$ is the base area. Equation (9) is the same as that given by Ward (1949) with extra terms due to the boundary layer and slight changes in notation.

Multiply equation (8) by U_∞ and subtract from equation (9) and we have

$$\bar{D} = \int_{S_1} p_\infty dS - \int_{S_2} \bar{\rho} U_\infty^2 \frac{\partial \bar{\phi}}{\partial r} \frac{\partial \bar{\phi}}{\partial x} dS - \int_{S_3} \left\{\bar{p} + \bar{\rho} U_\infty^2 \frac{\partial \bar{\phi}}{\partial x} \left(1 + \frac{\partial \bar{\phi}}{\partial x}\right)\right\} dS$$
$$- \int_{S_4+S_5} (p_b + \rho_b u_b^2 - \rho_b u_b U_\infty) dS - p_B S(D).$$

Now
$$\int_{S_1} p_\infty dS = \int_{S_3} p_\infty dS + \int_{S_4+S_5} p_\infty dS + p_\infty S(D).$$

Hence we have
$$\bar{D} = I_2 + I_3 + I_{45} - (p_B - p_\infty) S(D), \quad (10)$$

where
$$I_2 = -\int_{S_2} \bar{\rho} U_\infty^2 \frac{\partial \bar{\phi}}{\partial r} \frac{\partial \bar{\phi}}{\partial x} dS, \quad (11)$$

$$I_3 = -\int_{S_3} \{\bar{p} - p_\infty + \overline{\rho u^2} - \overline{\rho u} U_\infty\} dS = -\int_{S_3} K dS, \quad (12)$$

$$I_{45} = -\int_{S_4+S_5} \{p_b - p_\infty + \rho_b u_b^2 - \rho_b u_b U_\infty\} dS.$$

From now on we shall omit the last term in equation (10) for convenience. It may be inserted if necessary, that is if the body has a base.

3. The value of $I_3 + I_{45}$

We may write

$$I_3 = -\int_{S_3} K dS = -\int_{S_3+S_4} K dS + \int_{S_4} K dS. \quad (13)$$

In this equation we may give the part of the integrand which is integrated over S_4 any value we wish; we shall give $\bar{p}$, $\bar{\rho}$ and $\bar{u}$ their inviscid values due to flow over the distorted body. These are not their true values since S_4 is inside the boundary layer. The last term in equation (13) is equal to

$$\int_{S_4} (q\bar{c}_p + \bar{\rho}\bar{u}^2 - \bar{\rho}\bar{u}U_\infty)\, dS, \tag{14}$$

where we have written $\quad \bar{c}_p = \dfrac{\bar{p} - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \dfrac{\bar{p} - p_\infty}{q}.$

We denote values at the edge of the boundary layer by the subscript e. If in the expression (14) we replace $\bar{u}$ by $\bar{u}_e$ and $\bar{\rho}$ by $\bar{\rho}_e$ the error in the integrand is of order δ. Since S_4 is of order δ we may make this substitution in the integral (14) with an error of order δ^2. Hence we have

$$\begin{aligned} I_3 + I_{45} = -\int_{S_3+S_4} K\, dS + \int_{S_4+S_5} (\rho_e u_e^2 - \rho_e u_e U_\infty - \rho_b u_b^2 + \rho_b u_b U_\infty)\, dS \\ - \int_{S_5} (q\bar{c}_p + \bar{\rho}_e \bar{u}_e^2 - \bar{\rho}_e \bar{u}_e U_\infty)\, dS + \int_{S_4+S_5} (\bar{p} - p_b)\, dS. \end{aligned} \tag{15}$$

It is usual to assume that the change of pressure across the boundary layer is of order δ so that the last integral in equation (15) is of order δ^2 and may be neglected. This assumption may not be justified near to the trailing edge. Nevertheless, we shall make it for the time being, postponing further discussion on the matter to § 9.

With this assumption the sum of the last three integrals in equation (15), denoted by qN, may be written, on making use of equation (2)

$$qN = \int_{S_4+S_5} \{\rho_b u_b(\bar{u}_e - u_b) + \bar{u}_e'(\bar{\rho}_e \bar{u}_e - \rho_b u_b)\}\, dS - \int_{S_5} (q\bar{c}_p + \bar{\rho}\bar{u}_e \bar{u}_e')\, dS.$$

We note that

$$dS = (1 + \zeta/r)\, d\zeta\, d\sigma,$$

where ζ is measured normal to the section of the body by the plane at the trailing edge perpendicular to the free stream and r is the radius of curvature of the section by this plane, assumed to be of order unity. We define

$$\bar{\rho}_e \bar{u}_e^2 \Theta = \int_0^\delta \rho_b u_b(\bar{u}_e - u_b)(1 + \zeta/r)\, d\zeta = \int_0^\delta \rho_b u_b(\bar{u}_e - u_b)\, d\zeta + O(\delta^2),$$

$$\bar{\rho}_e \bar{u}_e^2 \Delta^* = \int_0^\delta (\bar{\rho}_e \bar{u}_e - \rho_b u_b)(1 + \zeta/r)\, d\zeta = \int_0^\delta (\bar{\rho}_e \bar{u}_e - \rho_b u_b)\, d\zeta + O(\delta^2).$$

We have also

$$\int_{S_5} \bar{c}_p\, dS = \int_D \bar{c}_p\, d\sigma \int_0^{\delta^*} (1 + \zeta/r)\, d\zeta = \int c_p \delta^*\, d\sigma + O(\delta^2).$$

We may replace the subscript e by D with error of order δ in each case. Thus we have, to order δ^2,

$$N = 2\int_D \frac{\rho_D u_D^2}{\rho_\infty U_\infty^2}\, \Theta\, d\sigma + 2\int_D \frac{\rho_D u_D u_D'}{\rho_\infty U_\infty^2} (\Delta^* - \delta^*)\, d\sigma - \int_D c_p \delta^*\, d\sigma.$$

Lighthill (1958) showed that if streamline co-ordinates ξ and η are taken such that the line elements along and perpendicular to the external streamlines are $h_\xi d\xi$ and $h_\eta d\eta$ respectively, then

$$\delta^* = \delta_\xi - \frac{1}{\rho_e U_\xi h_\eta}\frac{\partial}{\partial\eta}\int_0^\xi \rho_e U_\xi h_\xi \delta_\eta d\xi,$$

where $$\delta_\xi = \int_0^\delta \left(1-\frac{\rho u_\xi}{\rho_e U_\xi}\right) d\zeta, \quad \delta_\eta = \int_0^\delta \frac{\rho u_\eta}{\rho_e U_\xi} d\zeta$$

and U_ξ, u_ξ are the external and internal components of velocity in the streamline direction and u_η is the component normal to the streamlines. U_η is of course zero. It is possible to show that

$$\Delta^* = \delta_\xi + \delta_\eta \tan\alpha,$$

where α is the angle between the streamlines and the direction of flow at infinity, and so

$$\Delta^* - \delta^* = \frac{1}{\rho_e U_\xi h_\eta}\frac{\partial}{\partial\eta}\int_0^\xi \rho_e U_\xi h_\xi \delta_\eta d\xi + \delta_\eta \tan\alpha.$$

4. General theorem for drag

We have $$\bar{D} = I_2 - \int_{S_3+S_4} K\,dS + qN, \tag{16}$$

and this equation, excluding the last term, is the momentum-balance equation, with the same control surface as before, for inviscid flow past the distorted body, with base pressure equal to p_∞. In other words, the sum of the first two terms on the right-hand side of the equation is the drag of the displacement body, that is the integrated component of $\bar{p}-p_\infty$ over the surface.

Hence we have the general theorem: The drag of a body is equal to the drag of the displacement body together with a term qN, where

$$N = 2\int_D \frac{\rho_D u_D^2}{\rho_\infty U_\infty^2}\,\Theta\, d\sigma + 2\int_D \frac{\rho_D u_D u'_D}{\rho_\infty U_\infty^2}(\Delta^* - \delta^*)\, d\sigma - \int_D c_p \delta^* d\sigma. \tag{17}$$

If the body is slender with maximum semi-span s we may write this result in a simpler form. We have for a slender body

$$\frac{\rho_D}{\rho_\infty} = 1 - M^2\frac{u'_D}{U_\infty} + O(s^4\log^2 s), \tag{18}$$

$$c_p = \frac{p-p_\infty}{q} = -2\frac{u'_D}{U_\infty} - \frac{v_D^2+w_D^2}{U_\infty^2} + O(s^4\log^2 s), \tag{19}$$

and u'_D/U is of order $s^2\log s$. The second term in equation (17) is of order $\delta s^4\log^2 s$ if we suppose that u_η is of order s or smaller, so that δ_η is of order δs. The component u_η is usually small if the body is slender or thin (except perhaps when separation is approached). In such a case we may ignore this term and also replace δ^* by Δ^* in the last term. Hence we have for slender body

$$N = N' + E',$$

where $$N' = 2\int_D \Theta\left\{1 + \frac{u'_D}{U_\infty}(2+H-M^2)\right\} d\sigma, \tag{20}$$

$$E' = \int \frac{v_D^2+w_D^2}{U_\infty^2}\Delta^* d\sigma. \tag{21}$$

An alternative form for N' is

$$N' = 2\int_D \Theta\left(\frac{u_D}{U_\infty}\right)^{2+H-M^2} d\sigma. \tag{22}$$

We write N' in this form so as to compare it with the result of Squire & Young (1938).

For a slender body in supersonic flow the difference between the inviscid drag of the body and that of the displacement body is of order $\delta s^3 \log s$ or $\delta s^2 t \log s$. If we are content with a lower order of approximation in which such magnitudes may be ignored we may write the result: The drag of a body is equal to the inviscid drag of the body together with a term qN_1, where

$$N_1 = 2\int_D \Theta\, d\sigma. \tag{23}$$

In this case qN_1 is equal to the skin-friction drag.

If the body is thin but not slender and the cross flow is of order t we may write the result: The drag of a wing is equal to the drag of the displacement surface together with a term qN', where

$$N' = 2\int_D \Theta\left(\frac{u_D}{U_\infty}\right)^{2+H-M^2} d\sigma.$$

The E' term may here be dropped and the error is of order δt^2.

5. Drag of an axially symmetric body at zero incidence

If the body is axially symmetric it may have a blunt trailing edge (a circle) or a sharp trailing edge (a point). Suppose that the radius at the trailing edge is r_0.

In this case the integral I_{45}, which is of the form

$$\int_{S_4+S_5} f(r)\, dS$$

may be written

$$\int_0^\delta f(r)\, 2\pi(r_0+y)\, dy,$$

where y and δ are measured perpendicular to the axis of the body.

If we give θ and δ^* their usual definitions for axially symmetric flow, namely

$$\theta = \int_0^\delta \left(1+\frac{y}{r_0}\right)\frac{\rho_b u_b}{\rho_e u_e}\left(1-\frac{u_b}{u_e}\right) dy,$$

$$\delta^* = \int_0^\delta \left(1+\frac{y}{r_0}\right)\left(1-\frac{\rho_b u_b}{\rho_e u_e}\right) dy,$$

as given in Howarth (1953), we find

$$N = 4\pi r_0 \theta \frac{\rho_D u_D^2}{\rho_\infty U_\infty^2} - 2\pi r_0 \delta^* c_p, \tag{24}$$

and for a slender body

$$N' = 4\pi r_0 \theta \left(\frac{u_D}{U_\infty}\right)^{2+H-M^2}, \tag{25}$$

$$E' = 2\pi r_0 \delta^* \left(\frac{v_D}{U_\infty}\right)^2, \tag{26}$$

where v_D now represents the radial component of velocity of the inviscid flow at the trailing edge of the undistorted body.

If the body is pointed at the rear, θ and δ^* are infinite since r_0 is then zero, but $r_0\theta$ and $r_0\delta^*$ are finite.

If we are content with an error term of order $\delta s^3 \log s$ we find that

$$N_1 = 4\pi(r_0\theta), \tag{27}$$

and this gives a very simple form for the boundary-layer drag, as it is only necessary to find the value of $r_0\theta$ at the trailing edge of the body. Approximate methods of doing this are well known in both laminar and turbulent flows.

6. Drag of a slender thin wing with a sharp straight trailing edge in supersonic flow

We suppose that the trailing edge is at right angles to Ox, that Oy is parallel to it, and that $x = 1$ at the trailing edge.

In order to make use of the general theorem in § 4 we must find the drag of the distorted wing. For zero lift this was given approximately by Weber (1960) as an extension of Lighthill's formula (1956) to the case where the trailing edge has finite thickness. Using Weber's result we find that the total drag at zero lift is given by

$$\frac{\bar{D}}{q} = \frac{1}{2\pi}\int_0^1\int_0^1 \bar{S}''(x)\,\bar{S}''(x')\log\frac{1}{|x-x'|}\,dx\,dx' - \frac{\bar{S}'(1)}{\pi}\int_0^1 \bar{S}''(x)\log\frac{1}{1-x}\,dx$$
$$+\frac{\{\bar{S}'(1)\}^2}{2\pi}(\bar{k}+\Delta\bar{k}-\log\beta s)+E'+N', \tag{28}$$

where $$\beta^2 = M^2-1, \quad \bar{k} = \frac{\displaystyle\int_{-s}^{s}\int_{-s}^{s}\bar{\epsilon}(y)\,\bar{\epsilon}(y')\log\frac{2s}{|y-y'|}\,dy\,dy'}{\left\{\displaystyle\int_{-s}^{s}\bar{\epsilon}(y)\,dy\right\}^2},$$

$$\Delta\bar{k} \simeq -1{\cdot}3\boldsymbol{\delta}^*/s, \quad \bar{\epsilon}(y) = \partial\bar{z}(1,y)/\partial x,$$

$\boldsymbol{\delta}^*$ being the mean value of δ^* at the trailing edge.

We write $$\bar{S}(x) = S(x)+\Delta S(x), \quad \bar{D} = D+\Delta D,$$
$$\bar{\epsilon}(y) = \epsilon(y)+\Delta\epsilon(y), \quad \epsilon(y) = \partial z(1,y)/\partial x, \quad \Delta\epsilon(y) = \partial\delta^*(1,y)/\partial x,$$

and ignore squares and products of terms involving Δ. We find, after subtracting the inviscid drag D of the undistorted body, that the drag increment due to the boundary layer is given by

$$\frac{\Delta D}{q} = \frac{1}{\pi}\int_0^1\int_0^1 S''(x)\,\Delta S''(x)\log\frac{1}{|x-x'|}\,dx\,dx' - \frac{S'(1)}{\pi}\int_0^1 \Delta S''(x)\log\frac{1}{1-x}\,dx$$
$$-\frac{\Delta S'(1)}{\pi}\int_0^1 S''(x)\log\frac{1}{1-x}\,dx+\frac{S'(1)\,\Delta S'(1)}{\pi}(k'-\log\beta s)-\frac{1{\cdot}3\boldsymbol{\delta}^*}{2\pi s}\{S'(1)\}^2$$
$$+E'+N'+O(\delta s^3 t^2\log^2 s), \tag{29}$$

where $$k' = \frac{\displaystyle\int_{-s}^{s}\int_{-s}^{s}\epsilon(y)\,\Delta\epsilon(y')\log\frac{2s}{|y-y'|}\,dy\,dy'}{\displaystyle\int_{-s}^{s}\epsilon(y)\,dy\int_{-s}^{s}\Delta\epsilon(y)\,dy}. \tag{30}$$

It may be noted that the terms E' and $1{\cdot}3\boldsymbol{\delta}^*\{S'(1)\}^2/2\pi s$ are of order δst^2 whilst the other terms (excluding N') are of order $\delta s^2 t \log s$. The former may usually be ignored for a thin slender wing. In the example to follow E' and the $\boldsymbol{\delta}^*$ terms have been calculated; the contribution of each to the total drag is less than 0·1 %.

If we ignore these two terms there is no difficulty in extending the result to unsymmetrical wings at lift, provided that there is attached flow along the leading edges. All that is necessary to be done is to replace, in equation (30), the range $-s$ to s in all the integrals by D, where D is taken right round the trailing edge, top and bottom.

7. An example

In order to illustrate the method and to obtain some idea of the magnitudes involved we shall consider the delta wing with symmetrical rhombic sections and bi-convex centre section, for which (Weber 1957)

$$z(x,y) = \pm 2t(x-|y|/s)(1-x), \quad \epsilon(y) = 2t(1-|y|/s),$$
$$S'(x) = 4tsx(2-3x), \quad S''(x) = 8ts(1-3x),$$

where t is the maximum thickness.

We shall suppose that the boundary-layer flow is turbulent all over the wing, which is placed at zero incidence, and that it is the same as for flow over a flat plate of the same planform; also that for $M = 2$ the displacement thickness is approximately given by

$$\delta^* = 0{\cdot}095R^{-\frac{1}{5}}(x-|y|/s)^{\frac{4}{5}},$$

where R is the Reynolds number based on root chord. This gives

$$\Delta\epsilon(y) = L(1-|y|/s)^{-\frac{1}{5}}, \quad \Delta S'(x) = 5Lsx^{\frac{4}{5}},$$
$$\Delta S''(x) = 4Lsx^{-\frac{1}{5}}, \quad \Theta = \tfrac{5}{12}L(x-|y|/s)^{\frac{4}{5}},$$

where $$L = 0{\cdot}076R^{-\frac{1}{5}}.$$

The value of H has been taken from Young's (1953) curve, giving $H = 3$.

We now make use of equation (29), omitting E' and the term in $\boldsymbol{\delta}^*$ each of which contributes less than 0·1 % of the total drag. The integrals involved may all be evaluated but the numerical work is tedious and is omitted here. We shall write the increment of the drag coefficient over the inviscid value as ΔC_D, where

$$\Delta C_D = \Delta C_{D1} + N'/s,$$

C_{D1} being the drag of the displacement body; we write

$$N' = N_1 + N_2,$$

where $$N_1 = 2\int_D \Theta \, d\sigma, \quad N_2 = 2\int_D (2+H-M^2)\frac{u_D'}{U_\infty}\Theta \, d\sigma.$$

The following results are found:

$$\Delta C_{D1} = \left(\frac{Lts}{\pi}\right)(11{\cdot}99 + 20\log\beta s),$$
$$N_1 = \tfrac{50}{27}Ls, \quad N_2 = (40Lts^2/3\pi)(0{\cdot}655 - 1{\cdot}111\log\beta s),$$

where $$\beta^2 = M^2 - 1.$$

The inviscid drag coefficient of the wing is given by

$$C_D = (8t^2 s/\pi)(0{\cdot}605 - \log \beta s).$$

If the take $\beta s = 0{\cdot}4$, $s = 0{\cdot}23$, $t = 0{\cdot}05$, $R = 10^7$ we find

$$\Delta C_{D1}/C_D = -0{\cdot}031, \quad N_1/C_D s = 2{\cdot}48, \quad N_2/C_D s = 0{\cdot}100.$$

Thus we see that the main contribution to the drag increment comes from N_1, that is the part given by equation (23). This gives an increase of 248 % over the inviscid drag, whilst the other terms give a decrease of 3·1 % and an increase of 10·0 %, both of which are quite small compared to the total drag.

	C_{D1}	N_1	N_2	Inviscid
$R = 10^7$	$-0{\cdot}9$	69·7	3·1	28·1
$R = 4 \times 10^8$	$-0{\cdot}7$	54·4	2·4	43·9

TABLE 1. Contributions to drag as percentage of the total. $\beta s = 0{\cdot}4$.

If we take a Reynolds number of 4×10^8, such as might be appropriate for a wing of chord 200 ft. flying at Mach 2 at 55,000 ft., each of the contributions is halved. We obtain the results given in table 1, where we now express the various contributions as percentages of the total drag. This table suggests that in assessing the effect of the boundary layer on a slender wing at zero lift it may be sufficient to confine ourselves to the N_1 contribution and write

$$\Delta C_D = \frac{2}{A} \int_D \Delta \, d\sigma \tag{31}$$

taken round the trailing edge, where A is the area of the wing.

Varying the value of βs alters the results somewhat, and so we have worked out the effect of such variation, keeping $M = 2$. The results are shown in figure 3 for $R = 10^7$ and $R = 4 \times 10^8$. It is seen that the contributions from ΔC_{D1} and N_2 are small, so that the conclusion just given remains unaltered.

We have evaluated the integral N_1 for a case for which Θ was calculated by the method of Cooke (1961) at five points along the semi-span at the trailing edge. The wing was a 'delta' with 11 % thickness chord ratio and $\frac{4}{3}$ aspect ratio, the cross-sections being rhombic, having a different area distribution from that already considered. The Mach number was 2. The skin-friction drag of this model was not measured, but that of a model similar, but with three-quarters of the thickness, was found to be 0·00515. The change in thickness should not make much difference to the skin-friction drag at zero lift; indeed it is usually assumed to make no difference at all. The value of N_1 obtained by calculation is 0·00517.

8. Drag of a body with a swept sharp trailing edge

So far it has been supposed that the trailing edge lies in a plane normal to the direction of flow at infinity. If it is swept and sharp we may take the rear part of the control surface as that generated by lines normal to this edge and to the direction of flow at infinity. If at any point on this surface the local angle of sweep is Λ we must modify equations (8) and (9). Equation (10) becomes

$$\bar{D} = I_2 + J_3 + J_{45},$$

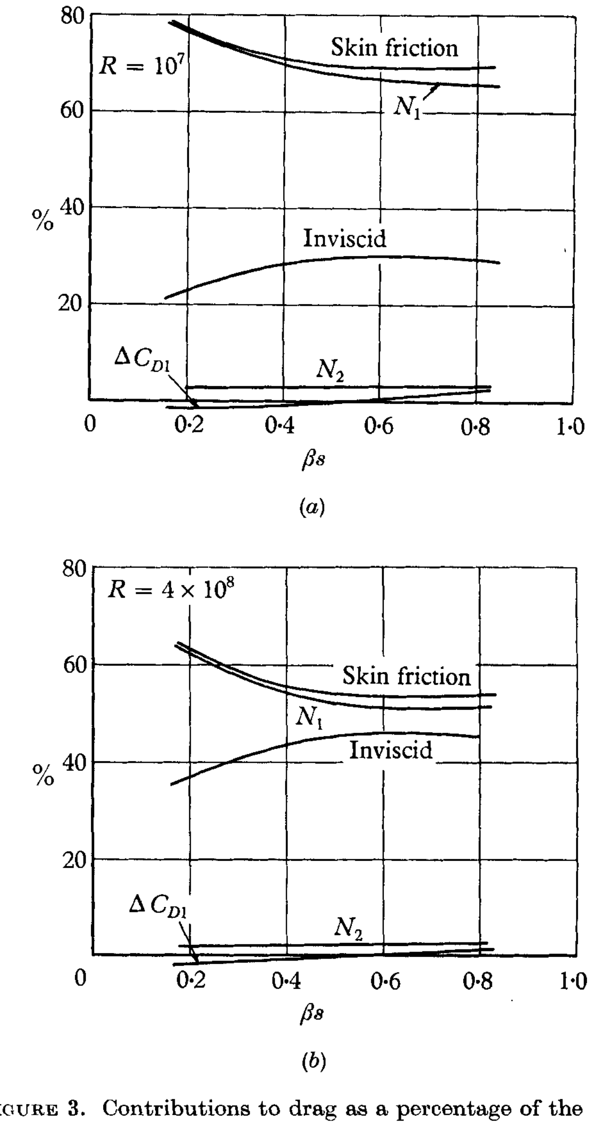

FIGURE 3. Contributions to drag as a percentage of the total. (*a*) $R = 10^7$; (*b*) $R = 4 \times 10^8$.

in which I_2 is the same as before, but

$$J_3 = -\int_{S_3} \{(\bar{p} - p_\infty)\cos\Lambda + \bar{\rho}(\bar{u} - U_\infty)(\bar{u}\cos\Lambda - \bar{v}\sin\Lambda)\}\, dS,$$

$$J_{45} = -\int_{S_4+S_5} \{(p_b - p_\infty)\cos\Lambda + \rho_b(u_b - U_\infty)(u_b\cos\Lambda - v_b\sin\Lambda)\}\, dS.$$

Analysis similar to that already given leads to the result

$$N = 2\int_D \left\{\frac{\rho_D u_D^2}{\rho_\infty U_\infty^2}\Theta + \frac{\rho_D u_D u_D'}{\rho_\infty U_\infty^2}(\Delta^* - \delta^*)\right\}\cos\Lambda\, d\sigma$$

$$-2\int_D \left\{\frac{\rho_D u_D^2}{\rho_\infty U_\infty^2}\Theta_{12} + \frac{\rho_D u_D u_D'}{\rho_\infty U_\infty^2}\left(\Delta'^* - \frac{v_D}{u_D}\delta^*\right)\right\}\sin\Lambda\, d\sigma - \int_D c_p \delta^* \cos\Lambda\, d\sigma,$$

where $$\rho_e u_e^2 \Theta_{12} = \int_0^\delta \rho v_b(u_e - u_b)\, d\zeta, \quad \rho_e u_e \Delta'^* = \int_0^\delta (\rho_e v_e - \rho_b v_b)\, d\zeta.$$

The line integrals are taken round the trailing edge as before. We can show that

$$\Delta'^* - (v_D/u_D)\,\delta^* = \tan\alpha(\delta_\xi - \delta^*) - \delta_\eta.$$

In the slender- or thin-wing case with cross flow of order s or t we find

$$N' = 2\int_D \Theta\left(\frac{u_D}{U_\infty}\right)^{2+H-M^2}\cos\Lambda\, d\sigma - 2\int_D \Theta_{12}\left(\frac{u_D}{U_\infty}\right)^{2-M^2}\sin\Lambda\, d\sigma,$$

$$E' = \int_D \frac{v_D^2 + w_D^2}{U_\infty^2}\Delta^* \cos\Lambda\, d\sigma,$$

and for a thin wing we may ignore E' as before.

As regards the value of Θ_{12} consideration of the velocity profiles to be expected (see, for instance, Cooke 1961) in turbulent flow (which generally occurs at the back of a body) leads to an approximate value for Θ_{12} in terms of Θ given by

$$\Theta_{12}/\Theta = 0{\cdot}5\gamma + \alpha,$$

where γ is the angle between streamlines and limiting streamlines and α is the angle between streamlines and the direction of flow at infinity, both measured at the trailing edge. For slender bodies and thin wings α is small and we have throughout assumed γ to be small. Hence at angles of sweep up to 60° or perhaps more we may probably ignore the term in Θ_{12} without serious error, though it will become of increasing importance if the angle of sweep is much more than this.

9. Discussion

We return first to the assumption made in § 3, which may be restated as implying that the mean value of p_b across the boundary layer differs from $\bar{p}$ by an amount of order δ. The assumption $p_b = \bar{p}$ is in fact usually made in boundary-layer theory, but this is open to doubt at the trailing edge of the wing. In two dimensions a more accurate expression for the pressure change is given by $\partial p/\partial\zeta = \rho_b \kappa_b u_b^2$, where κ_b is the curvature of the streamlines. At the trailing edge κ_b may be large and this will modify the result. In subsonic flow Spence (1954) found the pressure difference across a turbulent boundary layer at the trailing edge to be

$$\rho u_e^2 \tau/\omega\pi(H-1),$$

where τ is the trailing-edge angle. However, it is of course by no means certain that the analysis goes over into supersonic flow. If, however, it does apply in this case, our expression for $\bar{N}$ in equation (16) has an appearance of precision which is not justified, since the error in the last term in equation (16) is of the same order as the term itself. Thus until the matter of the pressure change across the boundary layer can be cleared up we are not justified in making a statement any more precise than that given in equation (23), for which the difficulty does not arise. This may not at present be a serious limitation in view of the fact that most applications involve thin wings and slender bodies, and of the fact that no great precision is yet possible in calculating turbulent boundary layers.

The difficulty may be partly avoided for thin wings or slender bodies by making the traverse at a short distance, say δ, upstream of the trailing edge, where the pressure change across the boundary layer may be expected to be small and where the upstream influence of the shock would be negligible. This would mean that the drag so calculated would in error by the amount of the drag on the small part of the wing cut off. If we denote the mean value of $c_p \tau$ over the distance δ by $(c_p \tau)_m$, where τ is the backwards slope of the surface, the pressure drag on the small part will be

$$-\delta \int_D (c_p \tau)_m \, d\sigma. \tag{32}$$

This, together with the skin friction of the small part (which is of order δ^r, where r is a little less than 2, say 1·8, and may be ignored) would then need to be added to the drag as calculated by the method of this paper. In the case of a slender body, c_p is of order $s^2 \log s$ and τ is of order s, so that the expression (32) will be of order $\delta s^4 \log s$, whilst E' is of order δs^3 and the second term in equation (20) for N' is of order $\delta s^3 \log s$. Hence if the drag were calculated by making the traverse a distance δ upstream and ignoring the correction term (32) the error in drag would be of order $\delta s^4 \log s$. Strictly one should now calculate the drag of the displacement surface with a piece of width δ cut off at the back. However, if the correction (32) is going to be ignored it would be better (and simpler) to take the drag of the full displacement surface as before, since doing this will help to neutralize the error made in ignoring the correction (32).

Consequently for a slender body, one would hope to obtain a good approximation by drag by making the traverse a distance δ upstream, calculating N' and E' by equations (20) and (21) and adding to the drag of the *full* displacement body; for a thin wing, equation (24) would also apply for such a modified traverse.

The conclusion that the extra drag may be given by the one equation (31) could possibly be tested by means of a boundary-layer traverse round the trailing edge. This would need to be ahead of the trailing-edge shock, but such a shock often seems to be a little behind the trailing edge; if the boundary layer there is turbulent the upstream influence of the shock is usually believed to be only about one or two boundary-layer thicknesses, so that there may be some hope of avoiding the effect of the shock and yet making the traverse quite near to the trailing edge.

In this connexion we may mention the work of Meyer (1957) who dealt with the general problem of determing drag in two-dimensional supersonic flow by means of a traverse behind the trailing edge. For the total drag it would be necessary to make a traverse over all of the area of the section of the nose shock by a normal plane through the trailing edge, and not merely inside the boundary layer. Meyer's study suggested that this would be difficult enough with present instrumental facilities even in two-dimensional flow; in three dimensions where possible cross flows may exist it might be very difficult indeed.

One difficulty not immediately brought to light in the above analysis is the effect of transition. In the example the boundary layer was assumed to be all turbulent. If it is laminar over part of the surface this will of course have its effect on Θ, reducing it and so reducing the contribution to the term N. There is no theoretical difficulty here. What *is* difficult is to assess the effect on what we have called ΔC_{D1}. There is rather a sudden drop on H at transition, which causes a corresponding drop in δ^*, since θ is supposed to be continuous at transition. This may result in a shock but even if it does not do this it raises a new difficulty. The displacement surface produced is not any longer amenable to a linear theory since its shape does not fulfil the assumptions of such a theory. This effect was first pointed out by Young & Kirkby (1955), who were able to deal with it satisfactorily in two dimensions. The difficulty does not affect the theory in § 4, but it makes the drag of the displacement surface difficult to calculate. It is possible that the method of Young & Kirkby could still be applied, though there may be error in applying two-dimensional simple-wave theory to this three-dimensional problem, especially if the transition front is swept, as it usually is.

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